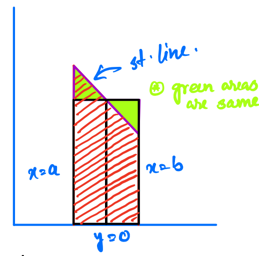
Approximating Areas:

Note:

Area = base × height

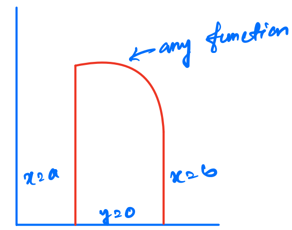
Easy



Area in red = Area of two rectangles.

So it's not easy, but by Simple Manipulation use rectangles it can be made

Easy



For this area we might be able to do similar.

Let's work with a particular example:

Find the area under the curve $y = x^2$ between $x = 0$ & $x = 4$.

y $f(x$

Here the base interval is $[0,4]$ (total length $4-0=4$)

Let's divide this interval in 4 equal pieces, then the length of each subinterval will be $\frac{4-0}{4} = \frac{4}{4} = 1 (= \Delta x)$

Then our 4 subintervals are:

$$[0,1],\ [1,2],\ [2,3]\ \&\ [3,4]$$

Now, we try to approximate the area using rectangles.

Left-End Point Approximation:

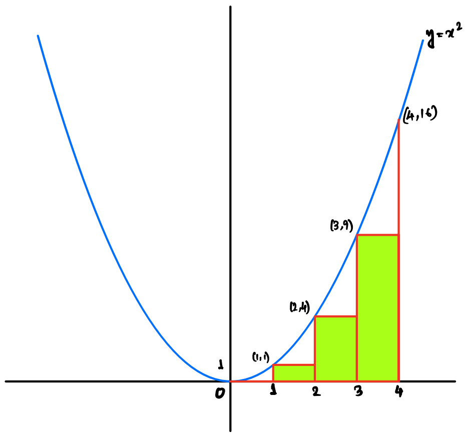
We pick the left end-points of the subintervals & the height of the rectangle will be the functional values of those left end-points of their corresponding rectangles.

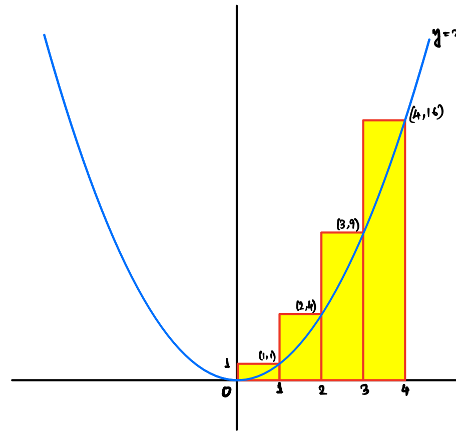
$$[0,1],\ [1,2],\ [2,3]\ \&\ [3,4]$$

$f(x) = x^2 \Rightarrow$ heights are $f(0) = 0$, $f(1) = 1$, $f(2) = 4$ & $f(3) = 9$

So, left end-point approximation for 4-subintervals is

$$L_4 = f(0)\cdot\Delta x + f(1)\cdot\Delta x + f(2)\cdot\Delta x + f(3)\cdot\Delta x$$
$$= [f(0)+f(1)+f(2)+f(3)]\cdot\Delta x = [0+1+4+9]\cdot 1 = 14$$





Right - End Point Approximation:

We pick the right end-points of the subintervals & the height of the rectangle will be the functional values of those right end-points of their corresponding rectangles.

$$[0,1],\ [1,2],\ [2,3]\ \&\ [3,4]$$

$f(x)=x^2 \Rightarrow$ heights are $f(1)=1$, $f(2)=4$, $f(3)=9$ & $f(4)=16$

So, right end-point approximation for 4-subintervals is

$$R_4 = f(1)\cdot\Delta x + f(2)\cdot\Delta x + f(3)\cdot\Delta x + f(4)\cdot\Delta x$$
$$= [f(1)+f(2)+f(3)+f(4)]\cdot\Delta x = [1+4+9+16]\cdot 1 = 30$$

Note: ① The actual area (A) is always between the left end-point approximation & right end-point approximation

Here $L_4 \leq A \leq R_4$

② The more subinterval you take, the more accurate area approximation you get, ie,

$$\lim_{n\to\infty}(\text{left end approx.}) = \lim_{n\to\infty}(\text{right end approx.}) = A.$$

So, for n subintervals $\Delta x = \frac{4-0}{n} = \frac{4}{n}$. & our subintervals are :

$$\left[0, \frac{4}{n}\right], \left[\frac{4}{n}, \frac{8}{n}\right], \left[\frac{8}{n}, \frac{12}{n}\right], \cdots\cdots, \left[\frac{4n-8}{n}, \frac{4n-4}{n}\right], \left[\frac{4n-4}{n}, 4\right]$$

Then

$$L_n = \left[f(0) + f\left(\frac{4}{n}\right) + f\left(\frac{8}{n}\right) + \cdots + f\left(\frac{4n-8}{n}\right) + f\left(\frac{4n-4}{n}\right)\right]\cdot \Delta x$$

$$= \left[0^2 + \left(\frac{4}{n}\right)^2 + \left(\frac{8}{n}\right)^2 + \cdots + \left(\frac{4n-8}{n}\right)^2 + \left(\frac{4n-4}{n}\right)^2\right]\cdot \frac{4}{n}.$$

$$= \left[\frac{4^2}{n^2}\cdot 0^2 + \frac{(4\cdot 1)^2}{n^2} + \frac{(4\cdot 2)^2}{n^2} + \cdots + \frac{(4(n-2))^2}{n^2} + \frac{(4(n-1))^2}{n^2}\right]\cdot \frac{4}{n}$$

$$= \left[\frac{4^2}{n^2}\cdot 0^2 + \frac{4^2}{n^2}\cdot 1^2 + \frac{4^2}{n^2}\cdot 2^2 + \cdots + \frac{4^2}{n^2}\cdot (n-2)^2 + \frac{4^2}{n^2}(n-1)^2\right]\cdot \frac{4}{n}.$$

$$= \frac{4^2}{n^2}\left[0^2 + 1^2 + 2^2 + \cdots + (n-2)^2 + (n-1)^2\right]\cdot \frac{4}{n}$$

$$= \frac{4^3}{n^3}\left[0^2 + 1^2 + 2^2 + \cdots + (n-2)^2 + (n-1)^2\right]$$

$$= \frac{4^3}{n^3}\cdot\left[1^2 + 2^2 + \cdots + (n-2)^2 + (n-1)^2\right]$$

Some known Sum formulas.

① $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

② $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

③ $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

$$= \frac{4^3}{n^3}\left[\frac{(n-1)(n-1+1)(2(n-1)+1)}{6}\right]$$

$$= \frac{4^3}{n^3}\left[\frac{(n-1)\,n\,(2n-1)}{6}\right]$$

Similarly, $R_n = \left[f\left(\frac{4}{n}\right) + f\left(\frac{8}{n}\right) + \cdots + f\left(\frac{4n-8}{n}\right) + f\left(\frac{4n-4}{n}\right) + f(4)\right] \cdot \Delta x$

$$= \frac{4^3}{n^3}\left[1^2 + 2^2 + \cdots + n^2\right]$$

$$= \frac{4^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

Now, $\lim_{n\to\infty} L_n = \lim_{n\to\infty}\left[\frac{4^3}{n^3} \cdot \frac{(n-1)\,n\,(2n-1)}{6}\right]$

$$= \lim_{n\to\infty}\left[\frac{4^3}{6} \cdot \frac{(n-1)(n)(2n-1)}{n \cdot n \cdot n}\right]$$

$$= \frac{4^3}{6} \cdot \lim_{n\to\infty}\left(1-\frac{1}{n}\right)(1)\cdot\left(2-\frac{1}{n}\right)$$

$$= \frac{4^3}{6} \cdot (1-0)\cdot(1)\cdot(2-0)$$

$$= \frac{4^3}{6} \cdot 2$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{4^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{4^3}{6} \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{n^3}$$

$n(n+1)(2n+1) \longrightarrow$ deg 3 with coeff. $1 \cdot 1 \cdot 2 = 2$

$n^3 \longrightarrow$ deg 3 with coeff. 1.

$$= \frac{4^3}{6} \cdot \frac{2}{1} = \frac{4^3}{6} \cdot 2$$

Now, since A is between L_n & R_n, for any n.

By $n \to \infty$, we have $A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \frac{4^3}{6} \cdot 2 = \frac{4^3}{3}$